

# Alternating Projections Methods for Discrete-time Stabilization of Quantum States

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**Abstract**—We study sequences (both cyclic and randomized) of idempotent completely-positive trace-preserving quantum maps, and show how they asymptotically converge to the intersection of their fixed point sets via alternating projection methods. We characterize the robustness features of the protocol against randomization and provide basic bounds on its convergence speed. The general results are then specialized to stabilizing entangled states in finite-dimensional multipartite quantum systems subject to a resource constraint, a problem of key interest for quantum information applications. We conclude by suggesting further developments, including techniques to enlarge the set of stabilizable states and ensure efficient, finite-time preparation.

## I. INTRODUCTION

Driving a quantum system to a desired state is a prerequisite for quantum control applications ranging from quantum chemistry to quantum computation [1]. An important distinction arises depending on whether the target state is to be obtained starting from a *known* initial state – in which case such knowledge may be leveraged to design a control law that effects a “one-to-one” state transfer; or, the initial state is *arbitrary* (possibly unknown) – in which case the dynamics must allow for “all-to-one” transitions. We shall refer to the latter as a *state preparation* task. The feasibility of these tasks, as well as the robustness and efficiency of the control protocol itself, are heavily influenced by the control resources that are permitted. Unitary control can only allow for transfer between states (represented by density operators), that are iso-spectral [2]. For both more general quantum state transfers, and for all state preparation tasks, access to *non-unitary* control (in the form of either coupling to an external reservoir or employing measurement and feedback) becomes imperative.

If a dissipative mechanism that “cools” the system to a known pure state is available, the combination of this all-to-one initialization step with state-controllable one-to-one unitary dynamics [3] is the simplest approach for achieving

pure-state preparation, with methods from optimal control theory being typically employed for synthesizing the desired unitary dynamics [1], [4]. In the same spirit, in the circuit model of quantum computation [5], preparation of arbitrary pure states is attained by initializing the quantum register in a known factorized pure state, and then implementing a sequence of unitary transformations (“quantum gates”) drawn from a universal set. Sampling from a mixed target state can be obtained by allowing for randomization of the applied quantum gates in conjunction with Metropolis-type algorithms [6]. Additional possibilities for state preparation arise if the target system is allowed to couple to a quantum auxiliary system, so that the pair can be jointly initialized and controlled, and the ancilla reset or traced over [7], [8]. For example, in [9] it is shown how sequential unitary coupling to an ancilla may be used to design a sequence of non-unitary transformations (“quantum channels”) on a target multi-qubit system that dissipatively prepare it in a matrix product state.

In the above-mentioned state-preparation methods, the dissipative mechanism is *fixed* (other than being turned on and off as needed), and the control design happens at the unitary level, directly on the target space or an enlarged one. A more powerful setting is to allow *dissipative control design* from the outset [10], [7], [11]. This opens up the possibility to synthesize all-to-one open-system dynamics that not only prepare the target state of interest but, additionally, leave it *invariant* throughout – that is, achieves *stabilization*, which is the task we focus on in this work. Quantum state stabilization has been investigated from different perspectives, including feedback design with classical [12], [13], [14], [15], [16], [17], [18], [19] and quantum [20], [21], [22], [23] controllers, as well as open-loop reservoir engineering techniques with both time-independent dynamics and switching control [24], [25], [26], [27], [28], [29], [30], [31], [32]. Most of this research effort, however, has focused on continuous-time models, with fewer studies addressing *discrete-time* quantum dynamics. With “digital” open-system quantum simulators being now experimentally accessible [33], [34], investigating quantum stabilization problems in discrete time becomes both natural and important. Thanks to the invariance requirement, “dissipative quantum circuits” bring distinctive advantages toward preparing pure or mixed target states *on-demand*, notably:

- While in a unitary quantum circuit or a sequential generation scheme, the desired state is only available at the completion of the full protocol, invariance of the target ensures that repeating a stabilizing protocol or even portions of it, will further maintain the system in the target state (if so desired), without disruption.

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- The order of the applied control operations need no longer be crucial, allowing for the target state to still be reached probabilistically (in a suitable sense), while incorporating robustness against the implementation order.
- If at a certain instant a wrong map is implemented, or some transient noise perturbs the dynamics, these unwanted effects can be re-absorbed without requiring active intervention or the whole preparation protocol having to be re-implemented correctly.

Discrete-time quantum Markov dynamics are described by sequences of quantum channels, namely, completely-positive, trace-preserving (CPTP) maps [35]. This give rise to a rich stability theory that can be seen as the non-commutative generalization of the asymptotic analysis of classical Markov chains, and that thus far has being studied in depth only in the time-homogeneous case [36], [37], including elementary feedback stabilizability and reachability problems [38], [39].

In this work, we show that *time-dependent* sequences of CPTP maps can be used to make their common fixed states the minimal asymptotically stable sets, which are reached by iterating cyclically a finite subsequence. The methods we introduce employ a finite number of idempotent CPTP maps, which we call *CPTP projections*, and can be considered a quantum version of *alternated projections methods*. The latter, stemming from seminal results by von Neumann [40] and extended by Halperin [41] and others [42], [43], are a family of (classical) algorithms that, loosely speaking, aim to select an element in the intersection of a number of sets that minimizes a natural (quadratic) distance with respect to the input. The numerous applications of such classical algorithms include estimation [44] and control [45] and, recently, specific tasks in quantum information, such as quantum channel construction [46]. In the context of quantum stabilization, we show that instead of working with the standard (Hilbert-Schmidt) inner product, it is natural to resort to a different inner product, a *weighted inner product* for which the CPTP projections become orthogonal, and the original results apply. When, depending on the structure of the fixed-point set, this strategy is not viable, we establish convergence by a different proof that does not directly build on existing alternating projection theorems. For all the proposed sequences, the order of implementation is not crucial, and convergence in probability is guaranteed even when the sequence is randomized, under very mild hypotheses on the distribution.

Section II introduces the models of interest, and recalls some key results regarding stability and fixed points of CPTP maps. Our general results on quantum alternating projections are presented in Section III, after proving that CPTP projections can be seen as orthogonal projections with respect to a weighted inner product. Basic bounds on the speed of convergence and robustness of the algorithms are discussed in Sections III-C and III-D, respectively. In Section IV we specialize these results to distributed stabilization of entangled states on multipartite quantum systems, where the robustness properties imply that the target can be reached by *unsupervised randomized applications* of dissipative quantum maps.

## II. PRELIMINARY MATERIAL

### A. Models and stability notions

We consider a finite-dimensional quantum system, associated to a Hilbert space  $\mathcal{H} \approx \mathbb{C}^d$ . Let  $\mathcal{B}(\mathcal{H})$  denote the space of linear bounded operators on  $\mathcal{H}$ , with  $\dagger$  being the adjoint operation. The state of the system at each time  $t \geq 0$  is a *density matrix* in  $\mathfrak{D}(\mathcal{H})$ , namely a positive-semidefinite, trace one matrix. Let  $\rho_0$  be the initial state. We consider *time inhomogenous* Markov dynamics, namely, sequences of CPTP maps  $\{\mathcal{E}_t\}$ , defining the state evolution for through the following dynamical equation:

$$\rho_{t+1} = \mathcal{E}_t(\rho_t), \quad t \geq 0. \quad (1)$$

Recall that a linear map  $\mathcal{E}$  is CPTP if and only if it admits an operator-sum representation (OSR) [35]:

$$\mathcal{E}(\rho) = \sum_k M_k \rho M_k^\dagger,$$

where the (Hellwig-Kraus) operators  $\{M_k\} \subset \mathcal{B}(\mathcal{H})$  satisfy  $\sum_k M_k^\dagger M_k = I$ . We shall assume that for all  $t > 0$  the map  $\mathcal{E}_t = \mathcal{E}_{j(t)}$  is chosen from a set of “available” maps, to be designed within the available control capabilities. In particular, in Section IV we will focus on locality-constrained dynamics. For any  $t \geq s \geq 0$ , we shall denote by

$$\mathcal{E}_{t,s} \equiv \mathcal{E}_{t-1} \circ \mathcal{E}_{t-2} \circ \dots \circ \mathcal{E}_s, \quad (\mathcal{E}_{t,t} = I),$$

the evolution map, or “propagator”, from  $s$  to  $t$ .

A set  $\mathcal{S}$  is *invariant* for the dynamics if  $\mathcal{E}_{t,s}(\tau) \in \mathcal{S}$  for all  $\tau \in \mathcal{S}$ . Define the distance of an operator  $\rho$  from a set  $\mathcal{S}$  as

$$d(\rho, \mathcal{S}) \equiv \inf_{\tau \in \mathcal{S}} \|\rho - \tau\|_1,$$

with  $\|\cdot\|_1$  being the trace norm. The following definitions are straightforward adaptations of the standard ones [47]:

*Definition 1:* (i) An invariant set  $\mathcal{S}$  is (uniformly) *simply stable* if for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $d(\tau, \mathcal{S}) < \delta$  ensures  $d(\mathcal{E}_{t,s}(\tau), \mathcal{S}) < \varepsilon$  for all  $t \geq s \geq 0$ . (ii) An invariant set  $\mathcal{S}$  is *globally asymptotically stable* (GAS) if it is simply stable and

$$\lim_{t \rightarrow \infty} d(\mathcal{E}_{t,s}(\rho), \mathcal{S}) = 0, \quad \forall \rho, s \geq 0. \quad (2)$$

Notice that, since we are dealing with finite-dimensional systems, convergence in any matrix norm is equivalent. Furthermore, since CPTP maps are trace-norm contractions [5], we have that simple stability is always guaranteed (and actually the distance is monotonically non-increasing):

*Proposition 1:* If a set  $\mathcal{S}$  is invariant for the dynamics  $\{\mathcal{E}_{t,s}\}_{t,s \geq 0}$ , then it is simply stable.

*Proof:* We have, for all  $t, s \geq 0$ :

$$\begin{aligned} d(\mathcal{E}_{t,s}(\rho), \mathcal{S}) &\leq d(\mathcal{E}_{t,s}(\rho), \mathcal{E}_{t,s}(\tau_{t,s}^*)) \\ &\leq d(\rho, \tau_{t,s}^*) \\ &= d(\rho, \mathcal{S}). \end{aligned}$$

The first inequality is true, by definition, for all  $\tau_{t,s} \in \mathcal{S}$ , and also on the closure  $\bar{\mathcal{S}}$ , thanks to continuity of  $\mathcal{E}_{t,s}$ ; the second holds due to contractivity of  $\mathcal{E}$ , and the last equality follows by letting  $\tau_{t,s}^* \equiv \arg \min_{\tau \in \mathcal{S}} \|\rho - \tau\|_1$ , where we can take the min since  $\mathcal{S}$  is closed and compact.  $\square$

### B. Fixed points of CPTP maps

We collect in this section some relevant results on the structure of fixed-point sets for a CPTP map  $\mathcal{E}$ , denoted by  $\text{fix}(\mathcal{E})$ . More details can be found in [48], [49], [36], [50].

Let  $\text{alg}(\mathcal{E})$  denote the  $\dagger$ -closed algebra generated by the operators in the OSR of  $\mathcal{E}$ , and  $\mathcal{A}'$  denote the commutant of  $\mathcal{A}$ , respectively. For unital CP maps,  $\text{fix}(\mathcal{E})$  is a  $\dagger$ -closed algebra,  $\text{fix}(\mathcal{E}) = \text{alg}(\mathcal{E})' = \text{fix}(\mathcal{E}^\dagger)$  [36], [50]. This implies that it admits a (Wedderburn) block decomposition [51]:

$$\text{fix}(\mathcal{E}) = \bigoplus_{\ell} \mathcal{B}(\mathcal{H}_{S,\ell}) \otimes I_{F,\ell}, \quad (3)$$

with respect to a Hilbert space decomposition:

$$\mathcal{H} = \bigoplus_{\ell} \mathcal{H}_{S,\ell} \otimes \mathcal{H}_{F,\ell}.$$

For general (not necessarily unital) CPTP maps the following holds [50], [36], [48]:

*Theorem 1 (Fixed-point sets, generic case):* Given a CPTP map  $\mathcal{E}$  which admits a full-rank fixed point  $\rho$ , we have

$$\text{fix}(\mathcal{E}) = \rho^{\frac{1}{2}} \text{fix}(\mathcal{E}^\dagger) \rho^{\frac{1}{2}}. \quad (4)$$

Moreover, with respect to the decomposition of  $\text{fix}(\mathcal{E}^\dagger) = \bigoplus_{\ell} \mathcal{B}(\mathcal{H}_{S,\ell}) \otimes I_{F,\ell}$ , the fixed state has the structure:

$$\rho = \bigoplus_{\ell} p_{\ell} \rho_{S,\ell} \otimes \tau_{F,\ell}, \quad (5)$$

where  $\rho_{S,\ell}$  and  $\tau_{F,\ell}$  are full-rank density operators of appropriate dimension, and  $p_{\ell}$  a set of convex weights.

This means that, given a CPTP map admitting a full-rank invariant state  $\rho$ , the fixed-point sets  $\text{fix}(\mathcal{E})$  is a  $\rho$ -distorted algebra, namely, an associative algebra with respect to a modified product (i.e.  $X \times_{\rho} Y = X \rho^{-1} Y$ ), with structure

$$\mathcal{A}_{\rho} = \bigoplus_{\ell} \mathcal{B}(\mathcal{H}_{S,\ell}) \otimes \tau_{F,\ell}, \quad (6)$$

where  $\tau_{F,\ell}$  are a set of density operators of appropriate dimension (the same for every element in  $\text{fix}(\mathcal{E})$ ). In addition, since  $\rho$  has the same block structure (5),  $\text{fix}(\mathcal{E})$  is clearly invariant with respect to the action of the linear map  $\mathcal{M}_{\rho,\lambda}(X) \equiv \rho^{\lambda} X \rho^{-\lambda}$  for any  $\lambda \in \mathbb{C}$ , and in particular for the modular group  $\{\mathcal{M}_{\rho,i\phi}\}$  [49]. The same holds for the fixed points of the dual dynamics. In [48], the following result has been proved using Takesaki's theorem, showing that commutativity with a modular-type operator is actually sufficient to ensure that a distorted algebra is a valid fixed-point set.

*Theorem 2: (Existence of  $\rho$ -preserving dynamics)* Let  $\rho$  be a full-rank density operator. A distorted algebra  $\mathcal{A}_{\rho}$ , such that  $\rho \in \mathcal{A}_{\rho}$ , admits a CPTP map  $\mathcal{E}$  such that  $\text{fix}(\mathcal{E}) = \mathcal{A}_{\rho}$  if and only if it is invariant for  $\mathcal{M}_{\rho,\frac{1}{2}}$ .

To our present aim, it is worth remarking that in the proof of the above result, a CPTP idempotent map is derived as the dual of a conditional expectation map, namely, the orthogonal projection onto the (standard) algebra  $\text{fix}(\mathcal{E}^\dagger)$ .

If the CPTP map  $\mathcal{E}$  does not admit a full rank invariant state, then it is possible to characterize the fixed-point set by first reducing to the support of the invariant states. This leads to the following structure theorem [50], [36], [48]:

*Theorem 3 (Fixed-point sets, general case):* Given a CPTP map  $\mathcal{E}$ , and a maximal-rank fixed point  $\rho$  with  $\tilde{\mathcal{H}} \equiv \text{supp}(\rho)$ , let  $\tilde{\mathcal{E}}$  denote the reduction of  $\mathcal{E}$  to  $\mathcal{B}(\tilde{\mathcal{H}})$ . We then have

$$\text{fix}(\mathcal{E}) = \rho^{\frac{1}{2}} (\ker(\tilde{\mathcal{E}}^\dagger) \oplus \mathbb{O}) \rho^{\frac{1}{2}}, \quad (7)$$

where  $\mathbb{O}$  is the zero operator on the complement of  $\tilde{\mathcal{H}}$ .

## III. ALTERNATING PROJECTION METHODS

### A. von Neumann-Halperin Theorem

Many of the ideas we use in this paper are inspired by a classical result originally due to von Neumann [40], and later extended by Halperin to multiple projectors:

*Theorem 4 (von Neumann-Halperin alternating projections):*

If  $\mathcal{M}_1, \dots, \mathcal{M}_r$  are closed subspaces in a Hilbert space  $\mathcal{H}$ , and  $P_{\mathcal{M}_j}$  are the corresponding orthogonal projections, then

$$\lim_{n \rightarrow \infty} (P_{\mathcal{M}_1} \dots P_{\mathcal{M}_r})^n x = Px, \quad \forall x \in \mathcal{H},$$

where  $P$  is the orthogonal projection onto  $\bigcap_{i=1}^r \mathcal{M}_i$ .

A proof for this theorem can be found in Halperin's original work [41]. Since then, the result has been refined in many ways, has inspired similar convergence results that use information projections [52] and, in full generality, projections in the sense of Bregman divergences [53], [42]. The applications of the results are manifold, especially in algorithms: while it is beyond the scope of this work to attempt a review, a good collection is presented in [43]. Some bounds on the convergence rate for the alternating projection methods can be derived by looking at the angles between the subspaces we are projecting on. We recall their definition and basic properties in Appendix A, see again [43] for more details.

### B. CPTP projections and orthogonality

We call an idempotent CPTP map, namely, one that satisfies  $\mathcal{E}^2 = \mathcal{E}$ , a CPTP projection. As any linear idempotent map,  $\mathcal{E}$  has only 0, 1 eigenvalues and maps any operator  $X$  onto the set of its fixed points,  $\text{fix}(\mathcal{E})$ . Recall that

$$\text{fix}(\mathcal{E}) = \bigoplus_{\ell} [\mathcal{B}(\mathcal{H}_{S,\ell}) \otimes \tau_{F,\ell}] \oplus \mathbb{O}_R, \quad (8)$$

for some Hilbert-space decomposition:

$$\mathcal{H} = \bigoplus_{\ell} (\mathcal{H}_{S,\ell} \otimes \mathcal{H}_{F,\ell}) \oplus \mathcal{H}_R, \quad (9)$$

where the last zero-block is not present if there exists a  $\rho > 0$  in  $\text{fix}(\mathcal{E})$ . We next give the structure of the CPTP projection associated to  $\text{fix}(\mathcal{E})$ : The result is essentially known (see e.g. [36], [49]) but we provide a short proof for completeness:

*Proposition 2:* Given a CPTP map  $\mathcal{E}$  with  $\rho$  a fixed point of maximal rank, a CPTP projection onto  $\mathcal{A}_{\rho} = \text{fix}(\mathcal{E})$  exists and is given by

$$\mathcal{E}_{\mathcal{A}_{\rho}}(X) = \lim_{T \rightarrow +\infty} \frac{1}{T} \sum_{i=0}^{T-1} \mathcal{E}^i(X). \quad (10)$$

If the fixed point  $\rho$  is full rank, then the CPTP projection onto  $\mathcal{A}_{\rho} = \bigoplus_{\ell} \mathcal{B}(\mathcal{H}_{S,\ell}) \otimes \tau_{F,\ell}$  is equivalently given by

$$\mathcal{E}_{\mathcal{A}_{\rho}}(X) = \bigoplus_{\ell} \text{Tr}_{F,\ell}(\Pi_{SF,\ell} X \Pi_{SF,\ell}) \otimes \tau_{F,\ell}, \quad (11)$$



where  $\Pi_{SF,\ell}$  is the orthogonal projection from  $\mathcal{H}$  onto the subspace  $\mathcal{H}_{S,\ell} \otimes \mathcal{H}_{F,\ell}$ .

*Proof:* Recalling that  $\|\mathcal{E}\|_\infty = 1$ , it is straightforward to prove that the limit in Eq. (10) exists and is a CPTP map. Furthermore, clearly  $\text{fix}(\mathcal{E}) \subseteq \text{fix}(\mathcal{E}_{\mathcal{A}_\rho})$ , and

$$\mathcal{E}_{\mathcal{A}_\rho} \mathcal{E} = \mathcal{E} \mathcal{E}_{\mathcal{A}_\rho} = \mathcal{E}_{\mathcal{A}_\rho}.$$

It follows that  $\mathcal{E}_{\mathcal{A}_\rho}^2 = \mathcal{E}_{\mathcal{A}_\rho}$ . On the other hand, it is immediate to verify that the right hand side of Eq. (11) is CPTP, has image equal to its fixed points  $\mathcal{A}_\rho = \text{fix}(\mathcal{E})$ , and is idempotent. Hence, it coincides with  $\mathcal{E}_{\mathcal{A}_\rho}$ .  $\square$

For a full-rank fixed-point set, CPTP projections are *not* orthogonal projections onto  $\text{fix}(\mathcal{E})$ , at least with respect to the Hilbert-Schmidt inner product. The proof is given in Appendix B. We are nonetheless going to show that  $\mathcal{E}_{\mathcal{A}}$  is an orthogonal projection with respect to a different inner product. This proves that the map in Eq. (11) is the unique CPTP projection onto  $\mathcal{A}_\rho$ . If the fixed-point set does not contains a full-rank state, Eq. (10) still defines a valid CPTP projection onto  $\text{fix}(\mathcal{E})$ ; however, this need not be unique. We will exploit this fact in the proof of Theorem 5, where we choose a particular one.

*Definition 2:* Let  $\xi$  be a positive-definite operator. (i) We define the  $\xi$ -inner product as

$$\langle X, Y \rangle_\xi \equiv \text{Tr}(X \xi Y); \quad (12)$$

(ii) We define the symmetric  $\xi$ -inner product as

$$\langle X, Y \rangle_{\xi,s} \equiv \text{Tr}(X \xi^{\frac{1}{2}} Y \xi^{\frac{1}{2}}). \quad (13)$$

It is straightforward to verify that both (12) and (13) are valid inner products.

We next show that  $\mathcal{E}_{\mathcal{A}}$  is an orthogonal projection with respect to (12) and (13), when  $\xi = \rho^{-1}$  for a full rank fixed point  $\rho$ . We will need a preliminary lemma. With  $W \equiv \bigoplus_i W_i$  we will denote an operator that acts as  $W_i$  on  $\mathcal{H}_i$ , for a direct-sum decomposition of  $\mathcal{H} = \bigoplus_i \mathcal{H}_i$ .

*Lemma 1:* Consider  $Y, W \in \mathcal{B}(\mathcal{H})$ , where  $W$  admits an orthogonal block-diagonal representation  $W = \bigoplus_\ell W_\ell$ . Then  $\text{Tr}(WY) = \sum_\ell \text{Tr}(W_\ell Y_\ell)$ , where  $Y_\ell = \Pi_\ell Y \Pi_\ell$ .

*Proof:* Let  $\Pi_\ell$  be the projector onto  $\mathcal{H}_\ell$ . Remembering that  $\sum_\ell \Pi_\ell = I$  and  $\Pi_\ell = \Pi_\ell^2$ , it follows that

$$\text{Tr}(X) = \sum_\ell \text{Tr}(\Pi_\ell X) = \sum_\ell \text{Tr}(\Pi_\ell X \Pi_\ell).$$

Therefore, we obtain:

$$\begin{aligned} \text{Tr}(WY) &= \text{Tr}\left(\sum_\ell \Pi_\ell \bigoplus_j W_j Y\right) = \sum_\ell \text{Tr}(\Pi_\ell W_\ell Y) \\ &= \sum_\ell \text{Tr}(\Pi_\ell W_\ell \Pi_\ell Y) = \sum_\ell \text{Tr}(W_\ell \Pi_\ell Y \Pi_\ell) \\ &= \sum_\ell \text{Tr}(W_\ell Y_\ell). \end{aligned} \quad \square$$

*Proposition 3:* Let  $\xi = \rho^{-1}$ , where  $\rho$  is a full-rank fixed state in  $\mathcal{A}_\rho$ , which is invariant for  $\mathcal{M}_{\rho, \frac{1}{2}}$ . Then  $\mathcal{E}_{\mathcal{A}_\rho}$  is an orthogonal projection with respect to the inner products in (12) and (13).

*Proof:* We already know that  $\mathcal{E}$  is linear and idempotent. In order to show that  $\mathcal{E}$  is an orthogonal projection, we need to show that it is self-adjoint relative to the relevant inner product. Let us consider  $\rho = \bigoplus_\ell \rho_\ell \otimes \tau_\ell$  and, as above:

$$\begin{aligned} X_\ell &= \Pi_{SF,\ell} X \Pi_{SF,\ell} = \sum_k A_{k,\ell} \otimes B_{k,\ell}, \\ Y_\ell &= \Pi_{SF,\ell} Y \Pi_{SF,\ell} = \sum_j C_{j,\ell} \otimes D_{j,\ell}. \end{aligned}$$

If we apply Lemma 1 to the operator

$$W = \mathcal{E}_{\mathcal{A}_\rho}(X) \rho^{-1} = \bigoplus_\ell ([\text{Tr}_{F,\ell}(X_\ell) \otimes \tau_\ell] (\rho_\ell^{-1} \otimes \tau_\ell^{-1})),$$

we obtain:

$$\begin{aligned} \langle \mathcal{E}(X), Y \rangle_\xi &= \text{Tr}(\mathcal{E}_{\mathcal{A}_\rho}(X) \rho^{-1} Y) \\ &= \text{Tr}\left(\bigoplus_\ell \text{Tr}_{F,\ell}(X_\ell) \otimes \tau_\ell (\rho_\ell^{-1} \otimes \tau_\ell^{-1}) Y_\ell\right) \\ &= \sum_{\ell,k,j} \text{Tr}([A_{k,\ell} \text{Tr}(B_{k,\ell}) \rho_\ell^{-1} \otimes I][C_{j,\ell} \otimes D_{j,\ell}]) \\ &= \sum_{\ell,k,j} \text{Tr}(B_{k,\ell}) \text{Tr}(A_{k,\ell} \rho_\ell^{-1} C_{j,\ell}) \text{Tr}(D_{j,\ell}). \end{aligned}$$

By similar calculation,

$$\begin{aligned} \langle X, \mathcal{E}(Y) \rangle_\xi &= \text{Tr}(X \rho^{-1} \mathcal{E}(Y)) \\ &= \sum_{\ell,k,j} \text{Tr}(B_{k,\ell}) \text{Tr}(A_{k,\ell} \rho_\ell^{-1} C_{j,\ell}) \text{Tr}(D_{j,\ell}). \end{aligned}$$

By comparison, we infer that  $\langle \mathcal{E}(X), Y \rangle_\xi = \langle X, \mathcal{E}(Y) \rangle_\xi$ . A similar proof can be carried over using the symmetric  $\xi$ -inner product of Eq. (13).  $\square$

We are now ready to prove the main results of this section. The first shows that the set of states with support on a target subspace can be made GAS by sequences of CPTP projections on larger subspaces that have the target as intersection.

*Theorem 5 (Subspace stabilization):* Let  $\mathcal{H}_j$ ,  $j = 1, \dots, r$ , be subspaces such that  $\bigcap_j \mathcal{H}_j \equiv \hat{\mathcal{H}}$ . Then there exists CPTP projections  $\mathcal{E}_1, \dots, \mathcal{E}_r$  onto  $\mathcal{B}(\mathcal{H}_j)$ ,  $j = 1, \dots, r$ , such that  $\forall \tau \in \mathcal{D}(\mathcal{H})$ :

$$\lim_{n \rightarrow \infty} (\mathcal{E}_r \dots \mathcal{E}_1)^n(\tau) = \mathcal{E}_{\mathcal{B}(\hat{\mathcal{H}})}(\tau), \quad (14)$$

where  $\mathcal{E}_{\mathcal{B}(\hat{\mathcal{H}})}$  is a CPTP projection onto  $\mathcal{B}(\hat{\mathcal{H}})$ .

*Proof:* We shall explicitly construct CPTP maps whose cyclic application ensures stabilization. Define  $P_j$  to be the projector onto  $\mathcal{H}_j$ , and the CPTP maps:

$$\mathcal{E}_j(\cdot) \equiv P_j(\cdot)P_j + \frac{P_j}{\text{Tr}(P_j)} \text{Tr}(P_j^\perp \cdot). \quad (15)$$

Consider  $\hat{P}$  the orthogonal projection onto  $\hat{\mathcal{H}}$  and the positive-semidefinite function  $V(\tau) = 1 - \text{Tr}(\hat{P} \tau)$ ,  $\tau \in \mathcal{B}(\mathcal{H})$ . The variation of  $V$ , when a  $\mathcal{E}_j$  is applied, is

$$\Delta V(\tau) \equiv V(\mathcal{E}_j(\tau)) - V(\tau) = -\text{Tr}[\hat{P}(\mathcal{E}_j(\tau) - \tau)] \equiv \Delta V_j(\tau).$$

If we show that this function is non-increasing along the trajectories generated by repetitions of the cycle of all maps, namely,  $\mathcal{E}_{\text{cycle}} \equiv \mathcal{E}_r \circ \dots \circ \mathcal{E}_1$ , the system is periodic thus its stability can be studied as a time-invariant one. Hence, by LaSalle-Krasowskii theorem [54], the trajectories (being all

bounded) will converge to the largest invariant set contained in the set of  $\tau$  such that on a cycle  $\Delta V_{\text{cycle}}(\tau) = 0$ . We next show that this set must have support *only* on  $\hat{\mathcal{H}}$ . If an operator  $\rho$  has support on  $\hat{\mathcal{H}}$ , it is clearly invariant and  $\Delta V(\rho) = 0$ . Assume now that  $\text{supp}(\tau) \not\subseteq \mathcal{H}_j$  for some  $j$ , that is,  $\text{Tr}(\tau P_j^\perp) > 0$ . By using the form of the map  $\mathcal{E}_j$  given in Eq. (15), we have

$$\begin{aligned} \Delta V_j(\tau) &= -\text{Tr}(\hat{P}(P_j \tau P_j)) - \text{Tr}(\tau P_j^\perp) \frac{\text{Tr}(\hat{P}(P_j))}{\text{Tr}(P_j)} \\ &+ \text{Tr}(\hat{P}\tau) \end{aligned}$$

The sum of the first and the third term in the above equation is zero since  $\hat{P} \leq P_j$ . The second term, on the other hand, is *strictly negative*. This is because: (i) we assumed that  $\text{Tr}(\tau P_j^\perp) > 0$ ; (ii) with  $\hat{P} \leq P_j$ , and  $\mathcal{E}_j(P_j)$  having the same support of  $P_j$  by construction, it also follows that  $\text{Tr}(\Pi \mathcal{E}_j(P_j)) > 0$ . This implies that  $\mathcal{E}_j$  either leaves  $\tau$  (and hence  $V(\tau)$ ) invariant, or  $\Delta V_j(\rho) < 0$ . Hence, each cycle  $\mathcal{E}_{\text{cycle}}$  is such that  $\Delta V_{\text{cycle}}(\tau) = \sum_{j=1}^r \Delta V_j(\tau) < 0$  for all  $\tau \notin \mathcal{D}(\hat{\mathcal{H}})$ . We thus showed that no state  $\tau$  with support outside of  $\hat{\mathcal{H}}$  can be in the attractive set for the dynamics. Hence, the dynamics asymptotically converges onto  $\mathcal{D}(\hat{\mathcal{H}})$  which is the only invariant set for all the  $\mathcal{E}_j$ .  $\square$

The second result shows that the a similar property holds for more general fixed-point sets, as long as they contain a full-rank state:

*Theorem 6 (Full-rank fixed-set stabilization):* Let the maps  $\mathcal{E}_1, \dots, \mathcal{E}_r$  be CPTP projections onto  $\mathcal{A}_i$ ,  $i = 1, \dots, r$ , and assume that  $\hat{\mathcal{A}} \equiv \bigcap_{i=1}^r \mathcal{A}_i$  contains a full-rank state  $\rho$ . Then  $\forall \tau \in \mathcal{D}(\mathcal{H})$ :

$$\lim_{n \rightarrow \infty} (\mathcal{E}_r \dots \mathcal{E}_1)^n(\tau) = \mathcal{E}_{\hat{\mathcal{A}}}(\tau), \quad (16)$$

where  $\mathcal{E}_{\hat{\mathcal{A}}}$  is the CPTP projection onto  $\hat{\mathcal{A}}$ .

*Proof:* Let us consider  $\xi = \rho^{-1}$ ; then  $\rho \in \hat{\mathcal{A}}$  implies that the maps  $\hat{\mathcal{E}}_i$  are all orthogonal projections with respect to the *same*  $\rho^{-1}$ -modified inner product (Propositions 2, 3). Hence, it suffice to apply von Neumann-Halperin, Theorem 4: asymptotically, the cyclic application of orthogonal projections onto subsets converges to the projection onto the intersection of the subsets; in our case, the latter is  $\hat{\mathcal{A}}$ .  $\square$

Together with Theorem 2, the above result implies that the intersection of fixed-point sets is still a fixed-point set of *some* map, as long as it contains a full-rank state:

*Corollary 1:* If  $\mathcal{A}_i$ ,  $i = 1, \dots, r$ , are  $\rho$ -distorted algebras, with  $\rho$  full rank, and are invariant for  $\mathcal{M}_{\rho, \frac{1}{2}}$ , then  $\hat{\mathcal{A}} = \bigcap_{i=1}^r \mathcal{A}_i$  is also a  $\rho$ -distorted algebra, invariant for  $\mathcal{M}_{\rho, \frac{1}{2}}$ .

*Proof:*  $\hat{\mathcal{A}}$  contains  $\rho$  and the previous Theorem ensures that a CPTP projection onto it exists. Then by Theorem 2 it is invariant for  $\mathcal{M}_{\rho, \frac{1}{2}}$ .  $\square$

Lastly, combining the ideas of the proof of Theorem 5 and 6, we obtain *sufficient* conditions for general fixed-point sets (that is, neither full algebras on a subspace nor containing a full-rank fixed-state).

*Theorem 7 (General fixed-point set stabilization):* Assume that the CPTP fixed-point sets  $\mathcal{A}_i$ ,  $i = 1, \dots, r$ , are such that

$\hat{\mathcal{A}} \equiv \bigcap_{i=1}^r \mathcal{A}_i$  satisfies

$$\text{supp}(\hat{\mathcal{A}}) = \bigcap_{i=1}^r \text{supp}(\mathcal{A}_i).$$

Then there exist CPTP projections  $\mathcal{E}_1, \dots, \mathcal{E}_r$  onto  $\mathcal{A}_i$ ,  $i = 1, \dots, r$ , such that  $\forall \tau \in \mathcal{D}(\mathcal{H})$ :

$$\lim_{n \rightarrow \infty} (\mathcal{E}_r \dots \mathcal{E}_1)^n(\tau) = \mathcal{E}_{\hat{\mathcal{A}}}(\tau), \quad (17)$$

where  $\mathcal{E}$  is a CPTP projection onto  $\hat{\mathcal{A}}$ .

*Proof:* To prove the claim, we explicitly construct the maps combining the ideas from the two previous theorems. Define  $P_j$  to be the projector onto  $\text{supp}(\mathcal{A}_j)$ , and the maps

$$\begin{aligned} \mathcal{E}_j^0(\cdot) &\equiv P_j(\cdot)P_j + \frac{P_j}{\text{Tr}(P_j)} \text{Tr}(P_j^\perp \cdot), \\ \mathcal{E}_j^1 &\equiv \mathcal{E}_{\mathcal{A}_j} \oplus \mathcal{I}_{\mathcal{A}_j^\perp}, \end{aligned}$$

where  $\mathcal{E}_{\mathcal{A}_j} : \mathcal{B}(\text{supp}(\mathcal{A}_j)) \rightarrow \mathcal{B}(\text{supp}(\mathcal{A}_j))$  is the unique CPTP projection onto  $\mathcal{A}_j$  (notice that on its own support  $\mathcal{A}_j$  includes a full-rank state), and  $\mathcal{I}_{\mathcal{A}_j^\perp}$  denotes the identity map on operators on  $\text{supp}(\mathcal{A}_j)^\perp$ . Now construct

$$\mathcal{E}_j(\cdot) \equiv \mathcal{E}_j^1 \circ \mathcal{E}_j^0(\cdot).$$

Since each map  $\mathcal{E}_j^1$  leaves the support of  $P_j$  invariant, the same Lyapunov argument of Theorem 5 shows that:

$$\text{supp}(\lim_{n \rightarrow \infty} (\mathcal{E}_r \dots \mathcal{E}_1)^n(\rho)) \subseteq \text{supp}(\mathcal{E}_{\mathcal{B}(\hat{\mathcal{H}})}(\tau)). \quad (18)$$

We thus have that the largest invariant set for a cycle of maps  $\mathcal{E}_r \dots \mathcal{E}_1$  has support equal to  $\hat{\mathcal{A}}$ , and by the discrete-time invariance principle [54], the dynamics converge to that.

Now notice that, since  $\hat{\mathcal{A}}$  is contained in each of the  $\mathcal{A}_j = \text{fix}(\mathcal{E}_j)$ , such is any maximum-rank operator in  $\hat{\mathcal{A}}$ , which implies (see e.g. Lemma 1 in [37]) that  $\text{supp}(\hat{\mathcal{A}})$  is an invariant subspace for each  $\mathcal{E}_j$ . Hence,  $\mathcal{E}_j$  restricted to  $\mathcal{B}(\text{supp}(\hat{\mathcal{A}}))$  is still CPTP, and by construction projects onto the elements of  $\mathcal{A}_j$  that have support contained in  $\text{supp}(\hat{\mathcal{A}})$ . Such a set, call it  $\hat{\mathcal{A}}_j$ , is thus a valid fixed-point set. By Theorem 6, we have that on the support of  $\hat{\mathcal{A}}$  the limit in Eq. (17) converges to  $\hat{\mathcal{A}}$ . This shows that the largest invariant set for the cycle is exactly  $\hat{\mathcal{A}}$ , hence the claim is proved.  $\square$

*Remark:* In order for the proposed quantum alternating projection methods to be effective, it is important that the relevant CPTP maps be sufficiently simple to evaluate and implement. Assuming that the map  $\mathcal{E}$  is easily achievable, it is useful to note that the projection map  $\mathcal{E}_{\mathcal{A}_\rho}$  defined in Eq. (10) may be approximated through iteration of a map  $\hat{\mathcal{E}}_\lambda \equiv (1 - \lambda)\mathcal{E} + \lambda\mathcal{I}$ , where  $\lambda \in (0, 1)$ . Since  $\hat{\mathcal{E}}_\lambda$  has 1 as the only eigenvalue on the unit circle [55], it is easy to show that  $\lim_{n \rightarrow \infty} \hat{\mathcal{E}}_\lambda^n = \mathcal{E}_{\mathcal{A}_\rho}$ ,  $\mathcal{E}_{\mathcal{A}_\rho} \approx \hat{\mathcal{E}}_\lambda^n$  for a sufficiently large number of iterations.

### C. Convergence rate

For practical applications, a relevant aspect of stabilizing a target set is provided by the rate of asymptotic convergence.

In our case, focusing for concreteness on state stabilization, the key to Theorems 6 and 7 (and to Theorem 9 that will be given in Sec. IV-C) is Halperin's alternating projection result.

Thus, if we are interested in the speed of convergence of stabilizing dynamics for a *full-rank state*  $\rho$ , this is just the speed of convergence of the classical alternating projection method. As mentioned, and as we recall in Appendix A, the rate is related to the angles between the subspaces. In fact, an upper bound for the distance decrease from the target attained in  $n$  repetitions of a cycle of maps  $\mathcal{E}_{\text{cycle}} = \mathcal{E}_r \circ \dots \circ \mathcal{E}_1$  is given in terms of the *contraction coefficient*  $c^{\frac{n}{2}}$ , where  $c \equiv 1 - \prod_{i=1}^{r-1} \sin^2 \theta_i$ , and  $\theta_i$  is the angle between  $\mathcal{A}_i$  and the intersection of the fixed points of the following maps. In particular, by Theorem 11 in the Appendix, it follows that a sufficient condition for *finite-time convergence* of iterated projections is given by  $c = 0$ , which is satisfied for example if  $c(\mathcal{A}_i, \mathcal{A}_j) = 0$  for all  $1 \leq i, j \leq r$ . That is, equivalently,

$$[\mathcal{A}_i \cap (\bigcap_{t=1}^r \mathcal{A}_t)^\perp] \perp_\rho [\mathcal{A}_j \cap (\bigcap_{t=1}^r \mathcal{A}_t)^\perp],$$

for every  $i, j = i + 1, \dots, r$ , where orthogonality is with respect to the  $\rho^{-1}$ -inner product, either symmetric or not. However, this condition is clearly *not* necessary.

For a *pure target state*  $\rho$ , a natural way to quantify the convergence rate is to consider the decrease of a suitable Lyapunov function. Given the form of the projection maps we propose, a natural choice is the same  $V$  we use in the proof of Theorem 5, namely,  $V(\tau) = 1 - \text{Tr}(\rho\tau)$ . The variation of  $V$ , when  $\mathcal{E}_{\text{cycle}}$  is applied, is

$$\Delta V(\tau) = -\text{Tr}[\rho(\mathcal{E}_{\text{cycle}}(\tau) - \tau)] < 0, \quad \forall \tau.$$

The contraction coefficient in the pseudo-distance  $V$  is then:

$$c = \max_{\tau \geq 0, \text{Tr}(\tau)=1, \text{Tr}(\tau\rho)=0} \Delta V(\tau).$$

In this way, we select the rate corresponding to the worst-case state with support orthogonal to the target (notice that maximization over all states would have just given zero).

#### D. Robustness with respect to randomization

While Theorem 5 and Theorem 6 require *deterministic* cyclic repetition of the CPTP projections, the order is not critical for convergence. Randomizing the order of the maps still leads to asymptotic convergence, albeit in probability. We say that an operator-valued process  $X(t)$  *converges in probability* to  $X^*$  if, for any  $\delta, \varepsilon > 0$ , there exists a time  $T > 0$  such that

$$\mathbb{P}[\text{Tr}((X(T) - X_*)^2) > \varepsilon] < \delta.$$

Likewise,  $X(t)$  *converges in expectation* if  $\mathbb{E}(\rho(t)) \rightarrow \rho^*$  when  $t \rightarrow +\infty$ . Establishing convergence in probability uses the following lemma, adapted from [56], a consequence of the second Borel-Cantelli lemma:

**Lemma 2 (Convergence in probability):** Consider a finite number of CPTP maps  $\{\mathcal{E}_j\}_{j=1}^M$ , and a (Lyapunov) function  $V(\rho)$ , such that  $V(\rho) \geq 0$  and  $V(\rho) = 0$  if and only if  $\rho \in \mathcal{S}$ , with  $\mathcal{S} \subset \mathcal{D}(\mathcal{H})$  some set of density operators. Assume, furthermore that:

- (i) For each  $j$  and state  $\rho$ ,  $V(\mathcal{E}_j(\rho)) \leq V(\rho)$ .

- (ii) For each  $\varepsilon > 0$  there exists a finite sequence of maps

$$\mathcal{E}_\varepsilon = \mathcal{E}_{j_K} \circ \dots \circ \mathcal{E}_{j_1}, \quad (19)$$

with  $j_\ell \in \{1, \dots, M\}$  for all  $\ell$ , such that  $V(\mathcal{E}_\varepsilon(\rho)) < \varepsilon$  for all  $\rho \notin \mathcal{S}$ .

Assume that the maps are selected at random, with independent probability distribution  $\mathbb{P}_t[\mathcal{E}_j]$  at each time  $t$ , and that there exists  $\varepsilon > 0$  for which  $\mathbb{P}_t[\mathcal{E}_j] > \varepsilon$  for all  $t$ . Then, for any  $\gamma > 0$ , the probability of having  $V(\rho(t)) < \gamma$  converges to 1 as  $t \rightarrow +\infty$ .

Using the above result, we can prove the following:

**Corollary 2:** Let  $\mathcal{E}_1, \dots, \mathcal{E}_r$  CPTP projections onto  $\mathcal{A}_i = \mathfrak{B}(\mathcal{H}_i)$ ,  $i = 1, \dots, r$ . Assume that at each step  $t \geq 0$  the map  $\mathcal{E}_{j(t)}$  is selected randomly from a probability distribution

$$\left\{ q_j(t) = \mathbb{P}[\mathcal{E}_{j(t)}] > 0 \mid \sum_j q_j(t) = 1 \right\},$$

and that  $q_j(t) > \epsilon > 0$  for all  $j$  and  $t \geq 0$ . For all  $\tau \in \mathfrak{D}(\mathcal{H})$ , let  $\tau(t) \equiv \mathcal{E}_{j(t)} \circ \dots \circ \mathcal{E}_{j(1)}(\tau)$ . Then  $\tau(t)$  converges in probability and in expectation to

$$\tau^* = \mathcal{E}_{\hat{\mathcal{A}}}(\tau),$$

where  $\mathcal{E}$  is the CPTP projection onto  $\hat{\mathcal{A}}$ .

**Proof:** Given Lemma 2, it suffices to consider  $V(\tau) \equiv 1 - \text{Tr}(\hat{\mathcal{P}}\tau)$ . It is non-increasing, and Theorem 6 also ensures that for every  $\varepsilon > 0$ , there exists a finite number of cycles of the maps that makes  $V(\tau) < \varepsilon$ .  $\square$

A similar result holds for the full-rank case:

**Corollary 3:** Let  $\mathcal{E}_1, \dots, \mathcal{E}_r$  CPTP projections onto  $\mathcal{A}_i$ ,  $i = 1, \dots, r$ , and assume that  $\hat{\mathcal{A}} = \bigcap_{i=1}^r \mathcal{A}_i$  contains a full-rank state  $\rho$ . Assume that at each step  $t \geq 0$  the map  $\mathcal{E}_{j(t)}$  is selected randomly from a probability distribution

$$\left\{ q_j(t) = \mathbb{P}[\mathcal{E}_{j(t)}] > 0 \mid \sum_j q_j(t) = 1 \right\},$$

and that  $q_j(t) > \epsilon > 0$  for all  $j$  and  $t \geq 0$ . For all  $\tau \in \mathfrak{D}(\mathcal{H})$ , let  $\tau(t) \equiv \mathcal{E}_{j(t)} \circ \dots \circ \mathcal{E}_{j(1)}(\tau)$ . Then  $\tau(t)$  converges in probability and in expectation to

$$\tau^* = \mathcal{E}_{\hat{\mathcal{A}}}(\tau),$$

where  $\mathcal{E}$  is the CPTP projection onto  $\hat{\mathcal{A}}$ .

**Proof:** Given the Lemma 2, it suffices to consider  $V(\tau) \equiv \langle (\tau - \tau^*), (\tau - \tau^*) \rangle_{\rho^{-1}}$ . It is non-increasing, and Theorem 6 ensures that for every  $\varepsilon > 0$  there exists a finite number of cycles of the maps that makes  $V(\tau) < \varepsilon$ .  $\square$

## IV. QUASI-LOCAL STATE STABILIZATION

### A. Locality notion

In this section we specialize to a multipartite quantum system consisting of  $n$  (distinguishable) subsystems, or “qudits”, defined on a tensor-product Hilbert space

$$\mathcal{H} \equiv \bigotimes_{a=1}^n \mathcal{H}_a, \quad a = 1, \dots, n, \quad \dim(\mathcal{H}_a) = d_a, \quad \dim(\mathcal{H}) = d.$$

In order to impose *quasi-locality constraints* on operators and dynamics on  $\mathcal{H}$ , we introduce *neighborhoods*. Following [27],

[28], [48], neighborhoods  $\{\mathcal{N}_j\}$  are subsets of indexes labeling the subsystems, that is,

$$\mathcal{N}_j \subsetneq \{1, \dots, n\}, \quad j = 1, \dots, K.$$

A *neighborhood operator*  $M$  is an operator on  $\mathcal{H}$  such that there exists a neighborhood  $\mathcal{N}_j$  for which we may write

$$M \equiv M_{\mathcal{N}_j} \otimes I_{\overline{\mathcal{N}_j}},$$

where  $M_{\mathcal{N}_j}$  accounts for the action of  $M$  on subsystems in  $\mathcal{N}_j$ , and  $I_{\overline{\mathcal{N}_j}} \equiv \bigotimes_{a \notin \mathcal{N}_j} I_a$  is the identity on the remaining ones. Once a state  $\rho \in \mathcal{D}(\mathcal{H})$  and a neighborhood structure are assigned on  $\mathcal{H}$ , *reduced neighborhood states* may be computed via partial trace as usual:

$$\rho_{\mathcal{N}_j} \equiv \text{Tr}_{\overline{\mathcal{N}_j}}(\rho), \quad \rho \in \mathcal{D}(\mathcal{H}), \quad j = 1, \dots, K, \quad (20)$$

where  $\text{Tr}_{\overline{\mathcal{N}_j}}$  indicates the partial trace over the tensor complement of the neighborhood  $\mathcal{N}_j$ , namely,  $\mathcal{H}_{\overline{\mathcal{N}_j}} \equiv \bigotimes_{a \notin \mathcal{N}_j} \mathcal{H}_a$ . A strictly “local” setting corresponds to the case where  $\mathcal{N}_j \equiv \{j\}$ , that is, each subsystem forms a distinct neighborhood.

Assume that some quasi-locality notion is *fixed* by specifying a set of neighborhoods,  $\mathcal{N} \equiv \{\mathcal{N}_j\}$ . A *CP map*  $\mathcal{E}$  is a *neighborhood map* relative to  $\mathcal{N}$  if, for some  $j$ ,

$$\mathcal{E} = \mathcal{E}_{\mathcal{N}_j} \otimes I_{\overline{\mathcal{N}_j}}, \quad (21)$$

where  $\mathcal{E}_{\mathcal{N}_j}$  is the restriction of  $\mathcal{E}$  to operators on the subsystems in  $\mathcal{N}_j$  and  $I_{\overline{\mathcal{N}_j}}$  is the identity map for operators on  $\mathcal{H}_{\overline{\mathcal{N}_j}}$ . An equivalent formulation can be given in terms of the OSR: that is,  $\mathcal{E}(\rho) = \sum_k M_k \rho M_k^\dagger$  is a neighborhood map relative to  $\mathcal{N}$  if there exists a neighborhood  $\mathcal{N}_j$  such that, *for all*  $k$ ,

$$M_k = M_{\mathcal{N}_j, k} \otimes I_{\overline{\mathcal{N}_j}}.$$

The reduced map on the neighborhood is then

$$\mathcal{E}_{\mathcal{N}_j}(\cdot) = \sum_k M_{\mathcal{N}_j, k} \cdot M_{\mathcal{N}_j, k}^\dagger.$$

Since the identity factor is preserved by sums (and products) of the  $M_k$ , it is immediate to verify that the property of  $\mathcal{E}$  being a neighborhood map is well-defined with respect to the freedom in the OSR [5].

**Definition 3:** (i) A state  $\rho$  is discrete-time *Quasi-Locally Stabilizable* (QLS) if there exists a sequence  $\{\mathcal{E}_t\}_{t \geq 0}$  of neighborhood maps such that  $\rho$  is GAS for the associated propagator  $\mathcal{E}_{t,s} = \mathcal{E}_{t-1} \circ \dots \circ \mathcal{E}_s$ , namely:

$$\mathcal{E}_{t,s}(\rho) = \rho, \quad \forall t \geq s \geq 0; \quad (22)$$

$$\lim_{t \rightarrow \infty} \|\mathcal{E}_{t,s}(\sigma), \rho\| = 0, \quad \forall \sigma \in \mathcal{D}(\mathcal{H}), \quad \forall s \geq 0. \quad (23)$$

(ii) The state is *QLS in finite time* (or finite-time QLS) if there exists a finite sequence of  $T$  maps whose propagator satisfies the invariance requirement of Eq. (22) and

$$\mathcal{E}_{T,0}(\sigma) = \rho, \quad \forall \sigma \in \mathcal{D}(\mathcal{H}). \quad (24)$$

**Remark:** With respect to the definition of quasi-locality that naturally emerges for continuous-time Markov dynamics [27], [28], [48], it is important to appreciate that constraining discrete-time dynamics to be QL in the above sense is more restrictive. In fact, even if a generator  $\mathcal{L}$  of a continuous-time (homogeneous) semigroup can be written as a sum of

neighborhood generators, namely,  $\mathcal{L} = \sum_k \mathcal{L}_k$ , the generated semigroup  $\mathcal{E}_t \equiv e^{\mathcal{L}t}$ ,  $t \geq 0$ , is *not*, in general, QL in the sense of Eq. (21) at any time. In some sense, one may think of the different noise components  $\mathcal{L}_1, \dots, \mathcal{L}_k$  of the continuous-time generator as acting “in parallel”. On the other hand, were the maps  $\mathcal{E}_j$  we consider in this paper each generated by some corresponding neighborhood generator  $\mathcal{L}_j$ , then by QL discrete-time dynamics we would be requesting that, on each time interval, a *single* noise operator is active, thus obtaining global switching dynamics [32] of the form

$$e^{\mathcal{L}_K T_K} \circ e^{\mathcal{L}_{K-1} T_{K-1}} \circ \dots \circ e^{\mathcal{L}_1 T_1}.$$

We could have requested each  $\mathcal{E}_t$  to be a convex combination of neighborhood maps acting on different neighborhoods, however it is not difficult to see that this case can be studied as the convergence in expectation for a randomized sequence. Hence, we are focusing on the *most restrictive definition of QL constraint* for discrete-time Markov dynamics. With respect to the continuous dynamics, however, we allow for the evolution to be time-inhomogeneous. Remarkably, we shall find a characterization of QLS pure states that is equivalent to the continuous-time case, when the latter dynamics are required to be *frustration free* (see Section IV-D).

#### B. Invariance conditions and minimal fixed point sets

In this section, we build on the invariance requirement of Eq. (22) to find *necessary* conditions that the discrete-time dynamics must satisfy in order to have a given state  $\rho$  as its unique and attracting equilibrium. These impose a certain minimal fixed-point set, and hence suggest a structure for the stabilizing dynamics.

Following [48], given an operator  $X \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ , with corresponding (operator) Schmidt decomposition  $X = \sum_j A_j \otimes B_j$ , we define its *Schmidt span* as:

$$\Sigma_A(X) \equiv \text{span}(\{A_j\}).$$

The Schmidt span is important because, if we want to leave an operator invariant with a neighborhood map, this also imposes the invariance of its Schmidt span. The following lemma, proved in [48], makes this precise:

**Lemma 3:** Given a vector  $v \in V_A \otimes V_B$  and  $M_A \in \mathcal{B}(V_A)$ , if  $(M_A \otimes \mathbb{I}_B)v = \lambda v$ , then  $(M_A \otimes \mathbb{I}_B)v' = \lambda v'$  for all  $v' \in \Sigma_A(v) \otimes V_B$ .

What we need here can be obtained by adapting this result to our case, specifically:

**Corollary 4:** Given a  $\rho \in \mathcal{D}(\mathcal{H}_{\mathcal{N}_j} \otimes \mathcal{H}_{\overline{\mathcal{N}_j}})$  and a neighborhood  $\mathcal{E} = \mathcal{E}_{\mathcal{N}_j} \otimes I_{\overline{\mathcal{N}_j}}$ , then  $\text{span}(\rho) \subseteq \text{fix}(\mathcal{E})$  implies

$$\Sigma_{\mathcal{N}_j}(\rho) \otimes \mathcal{B}(\mathcal{H}_{\overline{\mathcal{N}_j}}) \subseteq \text{fix}(\mathcal{E}_{\mathcal{N}_j}).$$

A Schmidt span need *not* be a valid fixed-point set, namely, a  $\rho$ -distorted algebra that is invariant for  $\mathcal{M}_{\rho, \frac{1}{2}}$ . In general, we need to further enlarge the QL fixed-point sets from the Schmidt span to suitable algebras. We discuss separately two relevant cases.

• *Pure states.*— Let  $\rho = |\psi\rangle\langle\psi|$  be a pure state and assume that, with respect to the factorization  $\mathcal{H}_{\mathcal{N}_j} \otimes \mathcal{H}_{\overline{\mathcal{N}_j}}$ ,



its Schmidt decomposition  $|\psi\rangle = \sum_k c_k |\psi_k\rangle \otimes |\phi_k\rangle$ . Let  $\mathcal{H}_{\mathcal{N}_j}^0 \equiv \text{span}\{|\psi_k\rangle\} = \text{supp}(\rho_{\mathcal{N}_j})$ . Then we have [48]:

$$\Sigma_{\mathcal{N}_j}(\rho) = \mathcal{B}(\mathcal{H}_{\mathcal{N}_j}^0). \quad (25)$$

In this case the Schmidt span is indeed a valid fixed-point set, and no further enlargement is needed. The *minimal fixed-point set* for neighborhood maps required to preserve  $\rho$  is thus  $\mathcal{F}_j \equiv \mathcal{B}(\mathcal{H}_{\mathcal{N}_j}^0) \otimes \mathcal{B}(\mathcal{H}_{\overline{\mathcal{N}}_j})$ . By construction, each  $\mathcal{F}_j$  contains  $\rho$ . Notice that their intersection is just  $\rho$  if and only if

$$\text{span}\{|\psi\rangle\} = \bigcap_j \mathcal{H}_{\mathcal{N}_j}^0 \otimes \mathcal{H}_{\overline{\mathcal{N}}_j} = \bigcap_j \mathcal{H}_j^0, \quad (26)$$

where we have defined  $\mathcal{H}_j^0 \equiv \mathcal{H}_{\mathcal{N}_j}^0 \otimes \mathcal{H}_{\overline{\mathcal{N}}_j}$ .

• *Full rank states.*— If  $\rho$  is a full-rank state, and  $W$  a set of operators, the *minimal fixed-point set generated by  $\rho$  and  $W$* , by Theorem 2, is the smallest  $\rho$ -distorted algebra generated by  $W$  which is invariant with respect to  $\mathcal{M}_{\rho, \frac{1}{2}}$ . Notice that, since  $\rho$  is full rank, its reduced states  $\rho_{\mathcal{N}_j}$  are also full rank. Denote by  $\text{alg}_\rho(W)$  the  $\dagger$ -closed  $\rho$ -distorted algebra generated by  $W$ . Call  $W_j \equiv \Sigma_{\mathcal{N}_j}(\rho)$ . The minimal fixed-point sets  $\mathcal{F}_{\rho_{\mathcal{N}_j}}(W_j)$  can then be constructed iteratively from  $\mathcal{F}_j^{(0)} \equiv \text{alg}_{\rho_{\mathcal{N}_j}}(W_j)$ , with the  $k$ -th step given by [48]:

$$\mathcal{F}_j^{(k+1)} \equiv \text{alg}_{\rho_{\mathcal{N}_j}}(\mathcal{M}_{\rho_{\mathcal{N}_j}, \frac{1}{2}}(\mathcal{F}_j^{(k)}), \mathcal{F}_j^{(k)}).$$

We keep iterating until  $\mathcal{F}_j^{(k+1)} = \mathcal{F}_j^{(k)} = \mathcal{F}_{\rho_{\mathcal{N}_j}}(W_j)$ . When that happens, define

$$\mathcal{F}_j \equiv \mathcal{F}_{\rho_{\mathcal{N}_j}}(\Sigma_{\mathcal{N}_j}(\rho)) \otimes \mathcal{B}(\mathcal{H}_{\overline{\mathcal{N}}_j}). \quad (27)$$

Since the  $\mathcal{F}_j$  are constructed to be the minimal sets for neighborhood maps that contain the given state and its corresponding Schmidt span, then clearly:

$$\text{span}(\rho) \subset \bigcap_j \mathcal{F}_j.$$

### C. Stabilizability under quasi-locality constraints

In the case of a pure target state, we can prove the following:

*Theorem 8 (QLS pure states):* A pure state  $\rho = |\psi\rangle\langle\psi|$  is QLS by discrete-time dynamics if and only if

$$\text{supp}(\rho) = \bigcap_j \mathcal{H}_j^0. \quad (28)$$

*Proof:* Given Corollary 4, any dynamics that make  $\rho$  QLS (and hence leaves it invariant) must consist of neighborhood maps  $\{\mathcal{E}_j\}$  with corresponding fixed points such that:

$$\mathcal{F}_k \subseteq \text{fix}(\mathcal{E}_j),$$

whenever  $\mathcal{E}_j$  is a  $\mathcal{N}_k$ -neighborhood map. If the intersection of the fixed-point sets is not unique, then  $\rho$  cannot be GAS, since there would be another state that is not attracted to it. Given Eq. (26), we have

$$\text{span}(\rho) = \bigcap_k \mathcal{F}_k \iff \text{supp}(\rho) = \bigcap_j \mathcal{H}_j^0,$$

which proves necessity. For sufficiency, we explicitly construct neighborhood maps whose cyclic application ensures stabilization. Define  $P_{\mathcal{N}_j}$  to be the projector onto  $\text{supp}(\rho_{\mathcal{N}_j})$ , and the CPTP maps:

$$\mathcal{E}_{\mathcal{N}_j}(\cdot) \equiv P_{\mathcal{N}_j}(\cdot)P_{\mathcal{N}_j} + \frac{P_{\mathcal{N}_j}}{\text{Tr}(P_{\mathcal{N}_j})} \text{Tr}(P_{\mathcal{N}_j}^\perp \cdot), \quad (29)$$

with  $\mathcal{E}_j \equiv \mathcal{E}_{\mathcal{N}_j} \otimes \text{Id}_{\overline{\mathcal{N}}_j}$ . Consider the positive-semidefinite function  $V(\tau) = 1 - \text{Tr}(\rho\tau)$ ,  $\tau \in \mathcal{B}(\mathcal{H})$ . The result then follows from Theorem 5.  $\square$

An equivalent characterization can be given for full-rank target states:

*Theorem 9 (QLS full-rank states):* A full-rank state  $\rho \in \mathcal{D}(\mathcal{H})$  is QLS by discrete-time dynamics if and only if

$$\text{span}(\rho) = \bigcap_k \mathcal{F}_k \quad (30)$$

*Proof:* As before, by contradiction, suppose that  $\rho_2 \in \bigcap_k \mathcal{F}_k$  exists, such that  $\rho_2 \neq \rho$ . This clearly implies that  $\rho$  cannot be GAS because there would exist another invariant state, which is not attracted to  $\rho$ . This proves necessity. Sufficiency derives from the alternating CPTP projection theorem. Specifically, let  $\mathcal{E}_{\mathcal{N}_k}$  be the CPTP projection onto  $\mathcal{F}_k$ , and

$$\mathcal{E}_k \equiv \mathcal{E}_{\mathcal{N}_k} \otimes \text{Id}_{\overline{\mathcal{N}}_k}.$$

By Theorem 6, we already know that for every  $\rho$ ,  $(\mathcal{E}_M \dots \mathcal{E}_1)^k(\rho) \rightarrow \bigcap_k \mathcal{F}_k$  for  $k \rightarrow \infty$ . Now, by hypothesis,  $\bigcap_k \mathcal{F}_k = \text{span}(\rho)$  and, being  $\rho$  the only (trace one) state in his own span,  $\rho$  is GAS.  $\square$

A set of *sufficient conditions*, stemming from Theorem 7, can be also derived in an analogous way for a general target state.

### D. Physical interpretation: discrete-time quasi-local stabilizability is equivalent to cooling without frustration

Consider a *quasi-local Hamiltonian*, that is,  $H = \sum_k H_k$ ,  $H_k = H_{\mathcal{N}_k} \otimes I_{\overline{\mathcal{N}}_k}$ .  $H$  is called a *parent Hamiltonian* for a pure state  $|\psi\rangle$  if it admits  $|\psi\rangle$  as a ground state, and it is called a *frustration-free (FF) Hamiltonian* if any global ground state is also a local ground state [57], that is,

$$\text{argmin}_{|\psi\rangle \in \mathcal{H}} \langle \psi | H | \psi \rangle \subseteq \text{argmin}_{|\psi\rangle \in \mathcal{H}} \langle \psi | H_k | \psi \rangle, \forall k.$$

Suppose that a target state  $|\psi\rangle$  admits a FF QL parent Hamiltonian  $H$  for which it is the *unique* ground state. Then, similarly to what has been done for continuous-time dissipative preparation [31], [27], the structure of  $H$  may be naturally used to derive a stabilizing discrete-time dynamics: it suffices to implement neighborhood maps  $M_k$  that stabilize the eigenspace associated to the minimum eigenvalue of each  $H_k$ . These can thought as maps that locally “cool” the system. In view of Theorem 8, it is easy to show that this condition is also necessary:

*Corollary 5:* A state  $\rho = |\psi\rangle\langle\psi|$  is QLS by discrete-time dynamics if and only if it is the unique ground state of a FF QL parent Hamiltonian.

*Proof:* Without loss of generality we can consider FF QL Hamiltonians  $H = \sum_k H_k$ , where each  $H_k$  is a projection.



Let  $\rho$  satisfy Eq. (28), which is equivalent to be QLS, and define  $H_k \equiv \Pi_{\mathcal{N}_k}^\perp \otimes I_{\overline{\mathcal{N}_k}}$ , with  $\Pi_{\mathcal{N}_k}^\perp$  being the orthogonal projector onto the orthogonal complement of the support of  $\rho_{\mathcal{N}_k}$ , that is,  $\mathcal{H}_{\mathcal{N}_k} \ominus \text{supp}(\rho_{\mathcal{N}_k})$ . Given Theorem 8,  $|\psi\rangle$  is the unique pure state in  $\bigcap_k \text{supp}(\rho_{\mathcal{N}_k} \otimes I_{\overline{\mathcal{N}_k}})$ , and thus the unique state in the kernel of all the  $H_k$ . Conversely, if a FF QL parent Hamiltonian exists, the kernels of the  $H_k$  satisfy the QLS condition and to each  $H_k$  we can associate a CPTP map as in (29) that projects onto its kernel.  $\square$

An equivalent characterization works for generic, full-rank target states, but we need to move from Hamiltonians to semi-group generators, while maintaining frustration-freeness in a suitable sense. If, as before,  $\mathcal{E}_t = e^{\mathcal{L}t}$  is the propagator arising from a time-invariant QL generator  $\mathcal{L}$ , we are interested in QL generators whose neighborhood components drive the system to a global equilibrium which is *also* a local equilibrium for each of them separately. That is, following [58], [48]:

**Definition 4:** A QL generator  $\mathcal{L} = \sum_j \mathcal{L}_j$ , where  $\mathcal{L}_j$  are neighborhood generators, is *Frustration Free* (FF) relative to a neighborhood structure  $\mathcal{N} = \{\mathcal{N}_j\}$  if

$$\rho \in \ker(\mathcal{L}) \iff \rho \in \ker(\mathcal{L}_j), \quad \forall j.$$

It is worth noting that a state which is invariant for all the local generators is always an equilibrium: the real requirement is that these states are *all* the equilibria. We then have:

**Proposition 4:** A pure or full-rank state  $\rho$  is discrete-time QLS if and only if it is QLS via FF continuous-time dynamics, that is, there exists a FF generator  $\mathcal{L}$  with respect to the same neighborhood structure  $\mathcal{N}$  such that

$$\lim_{t \rightarrow +\infty} e^{\mathcal{L}t} \rho_0 = \rho, \quad \forall \rho_0.$$

**Proof:** The claim follows from Theorems 7 and 8 in [48], which characterize the states that are unique fixed points of a FF generator as precisely the states that satisfy Eq. (30).  $\square$

**Remark:** Based on the above results, the conditions that guarantee either a pure or a full-rank state to be QLS in discrete time are the same that guarantee existence of a FF stabilizing generator in continuous time. We stress that if more general continuous-time generators are allowed, namely, frustration is permitted as in Hamiltonian-assisted stabilization [28], then the continuous-time setting can be more powerful. On the one hand, considering the stricter nature of the QL constraint for the discrete-time setting, this is not surprising. On the other hand, if Liouvillian is no longer FF, then the target is globally invariant for  $\mathcal{L}$  but no longer invariant for individual QL components  $\mathcal{L}_j$ , suggesting that a weaker (“stroboscopic”) invariance requirement could be more appropriate to “mimic” the effect of frustration in the discrete-time QL setting.

### E. Classes of discrete-time QLS states

Being the conditions for discrete-time QLS states the same as in continuous time with FF dynamics, we may conclude whether certain classes of states are QLS or not by exploiting the results already established in [27], [28], [48]. While we refer to the original references for additional detail and context,

some notable examples are summarized in what follows. Among *pure states*:

- 1) *Graph states* [59] (more generally, *stabilizer states* [5] that admit stabilizer group generators that are neighborhood operators) are discrete-time QLS. These states are entangled, and are a key resource for one-way quantum computation.
- 2) Certain, but not all, *Dicke states* are discrete-time QLS. Dicke states are symmetric with respect to subsystem permutations, and have a specified “excitation number” [60]. Dicke states exhibit entanglement properties that are, in some sense, robust: some entanglement is preserved even if some subsystems are measured or traced out. The  $n$ -qubit single-excitation Dicke state, also known as W-state,

$$|\psi_W^n\rangle \equiv \frac{1}{\sqrt{n}}(|100\dots 0\rangle + |010\dots 0\rangle + \dots + |000\dots 1\rangle),$$

fails to satisfy the conditions of Theorem 8, so  $\rho_W$  is *not* discrete-time QLS for non-trivial neighborhood structures (that is, unless there is a neighborhood that covers the whole network). On the other hand, for example, the two-excitation Dicke state on  $n = 4$  qubits,

$$|\psi_D^4\rangle \equiv \frac{|1100\rangle + |1010\rangle + |1001\rangle + |0110\rangle + |0101\rangle + |0011\rangle}{\sqrt{6}},$$

is QLS. A more general class of QLS Dicke states on qudits is presented in [48].

Among *full-rank states*:

- 1) *Commuting Gibbs states* are discrete-time QLS with respect to a suitable locality notion [48]. A Gibbs state represent the canonical thermal equilibrium state for a statistical system at temperature  $\beta^{-1}$ : if a chain of qudits is associated to a nearest-neighbor (NN) Hamiltonian  $H = \sum_k H_k$ , its Gibbs state is

$$\rho_\beta \equiv \frac{e^{-\beta H}}{\text{Tr}(e^{-\beta H})}.$$

If the Hamiltonian is commuting, namely,  $[H_j, H_k] = 0$  for all  $j, k$ , then  $\rho$  is QLS with respect to an *enlarged* QL notion, where  $\mathcal{N}_j^0$  contains all subsystems that belong to a NN neighborhood  $\mathcal{N}_k$  such that  $\mathcal{N}_k \cap \mathcal{N}_j \neq \emptyset$ . In analogy to the continuous-time case [58], this shows that Gibbs samplers based on QL discrete-time dissipative dynamics are also viable, at least in the commuting case.

- 2) *Certain mixtures of factorized and entangled states* are discrete-time QLS. For example, consider a 4-qubits system and the family of states parametrized by  $\epsilon \in (0, 1)$ :

$$\rho_\epsilon \equiv (1 - \epsilon) |\psi_D^4\rangle \langle \psi_D^4| + \epsilon |\text{GHZ}^4\rangle \langle \text{GHZ}^4|,$$

where  $|\text{GHZ}^n\rangle \equiv (|0\rangle^{\otimes n} + |1\rangle^{\otimes n})/\sqrt{2}$  denotes the maximally-entangled Greenberger-Horne-Zeilinger (GHZ) states on  $n$  qubits, and  $\mathcal{N}_1 = \{1, 2, 3\}$ ,  $\mathcal{N}_2 = \{2, 3, 4\}$ . This shows that we can stabilize states that are *arbitrarily close* to states that are provably not QLS, as the GHZ states [28], thereby achieving practical stabilization of the latter.

## V. CONCLUSIONS

We have introduced alternating projection methods based on sequences of CPTP projections, and used them in designing discrete-time stabilizing dynamics for entangled states in multipartite quantum systems subject to realistic quasi-locality constraints. When feasible, pursuing stabilization instead of preparation offers important advantages, including the possibility to retrieve the target state *on-demand*, at any (discrete) time after sufficient convergence is attained, since the invariance of the latter ensures that it is not ruined by subsequent maps. We show that the proposed methods are also suitable for distributed, randomized and unsupervised implementations on large networks. While the locality constraints we impose on the discrete-time dynamics are stricter, the stabilizable states are, remarkably, the same that are stabilizable for continuous-time frustration-free generators.

From a methodological standpoint, our results shed further light on the structure and *intersection of fixed-point sets* of CPTP maps, a structure of interest not only in control, but also in operator-algebraic approaches to quantum systems [61], quantum statistics [49] and quantum error correction theory [62], [63], [64]. In particular, we show that the intersection of fixed-point sets is *still* a fixed-point set, as long as it contains a full-rank state. In developing our results, we use both standard results from classical alternating projections and Lyapunov methods tailored to the positive linear maps at hand.

Towards applications, the proposed alternating projection methods are in principle suitable for implementation in digital open-quantum system simulators, such as demonstrated in proof-of-principle trapped-ion experiments [33]. Beside providing protocols for stabilizing relevant classes of pure entangled states, our methods point to an alternative approach for constructing quantum samplers using quasi-local resources.

Some developments of this line of research are worth highlighting. First, in order to extend the applicability of the proposed methods to more general classes of states, as well as to establish a tighter link to quantum error correction and dissipative code preparation, it is natural to look at discrete-time *conditional stabilization*, in the spirit of [28]. Notably, in [56], it has been shown that GHZ states and *all* Dicke states can be made conditionally asymptotically stable for QL discrete-time dynamics, with a suitable basin of attraction. Second, while we recalled some basic classical bounds on the convergence speed, that apply to the stabilization of full-rank states, their geometric nature makes it hard to obtain useful insight from them. A more intuitive approach to convergence speed and its optimization, following e.g. [26], [37], may offer a more promising venue in that respect. It has also been recently shown that linear Lyapunov functions can not only be used to prove convergence, but also provide sharp bounds on the convergence speed in continuous-time dynamics [65]. It would be interesting to extend the analysis to the non-homogeneous, discrete-time cases considered in this work. Lastly, the characterization of physically relevant scenarios in which *finite-time stabilization* is possible under locality constraints is a challenging open problem, which we plan to address elsewhere [66].

## ACKNOWLEDGEMENTS

It is a pleasure to acknowledge stimulating discussions on the topics of this work with A. Ferrante e L. Finesso. F.T. is especially grateful to V. Umanità and E. Sasso for pointing him towards Takesaki's theorem. Work at Dartmouth was supported by the National Science Foundation through grant No. PHY-1620541.

## APPENDIX

### A. Angles between subspaces

Define the function  $\arccos : [-1, 1] \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$ . We will use only the elements in interval  $[0, 1]$ . Then the *angle*  $\theta(\mathcal{M}, \mathcal{N})$  between two closed subspaces  $\mathcal{M}$  and  $\mathcal{N}$  of  $\mathcal{H}$  is an element of  $[0, \frac{\pi}{2}]$ . We have the following:

*Definition 5:* The cosine  $c(\mathcal{M}, \mathcal{N})$  between two closed subspaces  $\mathcal{M}$  and  $\mathcal{N}$  of  $\mathcal{H}$  is given by

$$c(\mathcal{M}, \mathcal{N}) \equiv \sup \left\{ |\langle x, y \rangle| : x \in \mathcal{M} \cap (\mathcal{M} \cap \mathcal{N})^\perp, \|x\| \leq 1, y \in \mathcal{N} \cap (\mathcal{M} \cap \mathcal{N})^\perp, \|y\| \leq 1 \right\}.$$

Then the angle is given by:

$$\theta(\mathcal{M}, \mathcal{N}) = \arccos(c(\mathcal{M}, \mathcal{N})).$$

Some consequences of the above definitions are the following:

- 1)  $0 \leq c(\mathcal{M}, \mathcal{N}) \leq 1$ ;
- 2)  $c(\mathcal{M}, \mathcal{N}) = c(\mathcal{N}, \mathcal{M})$ ;
- 3)  $c(\mathcal{M}, \mathcal{N}) = \|P_{\mathcal{M}}P_{\mathcal{N}} - P_{\mathcal{M} \cap \mathcal{N}}\| = \|P_{\mathcal{M}}P_{\mathcal{N}}P_{(\mathcal{M} \cap \mathcal{N})^\perp}\|$ .

We next state the result that gives the exact rate in case of projection onto two subspaces [43]:

*Theorem 10:* In the norm induced by the inner product, and for each  $n$ , the following equality holds:

$$\|(P_{\mathcal{M}_2}P_{\mathcal{M}_1})^n - P_{\mathcal{M}_1 \cap \mathcal{M}_2}\| = c(\mathcal{M}_1, \mathcal{M}_2)^{2n-1}.$$

In case of alternating projections on the intersection of more than two subspaces, an exact expression is no longer available, however an upper bound may be given [43]:

*Theorem 11:* For each  $i = 1, 2, \dots, r$ , let  $\mathcal{M}_i$  be a closed subspace of  $\mathcal{H}$ . Then, for each  $x \in \mathcal{H}$ , and for any integer  $n \geq 1$  it holds:

$$\|(P_{\mathcal{M}_r} \dots P_{\mathcal{M}_1})^n x - P_{\bigcap_{i=1}^r \mathcal{M}_i} x\| \leq c^{\frac{n}{2}} \|x - P_{\bigcap_{i=1}^r \mathcal{M}_i} x\|,$$

where the contraction coefficient

$$c = 1 - \prod_{i=1}^{r-1} \sin^2 \theta_i,$$

and  $\theta_i$  is the angle between  $\mathcal{M}_i$  and  $\bigcap_{j=i+1}^r \mathcal{M}_j$ .

### B. Non-orthogonality of $\mathcal{E}_{\mathcal{A}}$ with respect to the Hilbert-Schmidt inner product

Let us decompose a full-rank fixed point set  $\mathcal{A}_\rho = \bigoplus_\ell \mathcal{A}_\ell = \bigoplus_\ell \mathcal{B}(\mathcal{H}_{S,\ell}) \otimes \tau_\ell$ , (where  $\tau_\ell \equiv \tau_{F,\ell}$ ). By definition, the orthogonal projection of  $X$  onto  $\mathcal{A}_i$  is given by

$$P_{\mathcal{A}}(X) \equiv \sum_{\ell, i} \langle \sigma_{\ell, i} \otimes \tau_\ell, X \rangle_{HS} \sigma_{\ell, i} \otimes \tau_\ell,$$

where  $\sigma_{\ell,i} \otimes \tau_{\ell}$  is an orthonormal basis for  $\mathcal{A}_{\ell}$ . Note that the outcome only depends on the restrictions of  $X$  to the supports of the  $\mathcal{A}_{\ell}$ . Hence, decompose  $X \equiv \sum_{\ell} X_{\ell} + \Delta X$ , where  $X_{\ell} = \Pi_{SF,\ell} X \Pi_{SF,\ell}$ , and further decompose  $X_{\ell} \equiv \sum_k A_{\ell,k} \otimes B_{\ell,k}$ , so we can write:

$$\begin{aligned} P_{\mathcal{A}}(X) &= \bigoplus_i \sum_{j,\ell} \left( \sum_k \text{Tr}[(\sigma_j \otimes \tau_{\ell})(A_{\ell,k} \otimes B_{\ell,k})] \sigma_j \otimes \tau_{\ell} \right) \\ &= \bigoplus_{\ell} \sum_{j,\ell} \left( \text{Tr}[\sigma_j \sum_k (A_{\ell,k} \text{Tr}(\tau_{\ell} B_{\ell,k}))] \sigma_j \otimes \tau_{\ell} \right). \end{aligned}$$

By comparing the latter equation with Eq. (11), we have that  $P_{\mathcal{A}} = \mathcal{E}_{\mathcal{A}}$  if and only if  $\sum_k (A_{\ell,k} \text{Tr}(\tau_j B_k)) = \text{Tr}_{F,\ell}(X_{\ell})$ , which is equivalent to request that  $\tau_j = \lambda_{\ell} I$ . Hence, unless  $\mathcal{A}_{\rho}$  contains the completely mixed state,  $\mathcal{E}_{\mathcal{A}}$  in Eq. (11) is not an orthogonal projection with respect to the Hilbert-Schmidt inner product.  $\square$

## REFERENCES

- [1] C. Altafini and F. Ticozzi, “Modeling and control of quantum systems: An introduction,” *IEEE Trans. Aut. Contr.*, vol. 57, no. 8, pp. 1898–1917, 2012.
- [2] S. G. Schirmer, A. I. Solomon, and J. V. Leahy, “Criteria for reachability of quantum states,” *J. Phys. A*, vol. 35, pp. 8551–8562, 2002.
- [3] D. D’Alessandro, *Introduction to Quantum Control and Dynamics*, ser. Applied Mathematics & Nonlinear Science. Chapman & Hall/CRC, 2007.
- [4] K. Rojan, D. M. Reich, I. Dotsenko, J.-M. Raimond, C. P. Koch, and G. Morigi, “Arbitrary-quantum-state preparation of a harmonic oscillator via optimal control,” *Phys. Rev. A*, vol. 90, p. 023824, 2014.
- [5] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Information*. Cambridge University Press, Cambridge, 2002.
- [6] K. Temme, T. J. Osborne, K. G. Vollbrecht, D. Poulin, and F. Verstraete, “Quantum Metropolis sampling,” *Nature*, vol. 471, pp. 87–90, 2010.
- [7] S. Lloyd and L. Viola, “Engineering quantum dynamics,” *Phys. Rev. A*, vol. 65, pp. 010101:1–4, 2001.
- [8] F. Ticozzi and L. Viola, “Quantum resources for purification and cooling: fundamental limits and opportunities,” *Sci. Rep.*, vol. 4, pp. 5192/1–7, 2014.
- [9] C. Schön, E. Solano, F. Verstraete, J. I. Cirac, and M. M. Wolf, “Sequential generation of entangled multiqubit states,” *Phys. Rev. Lett.*, vol. 95, p. 110503, 2005.
- [10] J. F. Poyatos, J. I. Cirac, and P. Zoller, “Quantum reservoir engineering with laser cooled trapped ions,” *Phys. Rev. Lett.*, vol. 77, no. 23, pp. 4728–4731, 1996.
- [11] R. Wu, A. Pechen, C. Brif, and H. Rabitz, “Controllability of open quantum systems with Kraus-map dynamics,” *J. Phys. A*, vol. 40, no. 21, pp. 5681–5693, 2007.
- [12] V. P. Belavkin, “Towards the theory of control in observable quantum systems,” *Automatica and Remote Control*, vol. 44, pp. 178–188, 1983.
- [13] H. M. Wiseman, “Quantum theory of continuous feedback,” *Phys. Rev. A*, vol. 49, no. 3, pp. 2133–2150, 1994.
- [14] A. C. Doherty and K. Jacobs, “Feedback control of quantum systems using continuous state estimation,” *Phys. Rev. A*, vol. 60, no. 4, pp. 2700–2711, 1999.
- [15] R. van Handel, J. K. Stockton, and H. Mabuchi, “Feedback control of quantum state reduction,” *IEEE Trans. Aut. Contr.*, vol. 50, no. 6, pp. 768–780, 2005.
- [16] M. Mirrahimi and R. V. Handel, “Stabilizing feedback controls for quantum systems,” *SIAM J. Control Optim.*, vol. 46, pp. 445–467, April 2007.
- [17] C. Sayrin, I. Dotsenko, X. Zhou, B. Peaudecerf, T. Rybarczyk, S. Gleyzes, P. Rouchon, M. Mirrahimi, H. Amini, M. Brune, J.-M. Raimond, and S. Haroche, “Real-time quantum feedback prepares and stabilizes photon number states,” *Nature*, vol. 477, pp. 73–77, 2011.
- [18] F. Ticozzi and L. Viola, “Quantum Markovian subsystems: Invariance, attractivity and control,” *IEEE Trans. Aut. Contr.*, vol. 53, no. 9, pp. 2048–2063, 2008.
- [19] —, “Analysis and synthesis of attractive quantum Markovian dynamics,” *Automatica*, vol. 45, pp. 2002–2009, 2009.
- [20] H. M. Wiseman and G. J. Milburn, “Quantum theory of optical feedback via homodyne detection,” *Phys. Rev. Lett.*, vol. 70, no. 5, pp. 548–551, 1993.
- [21] J. Gough and M. R. James, “Quantum feedback networks: Hamiltonian formulation,” *Comm. Math. Phys.*, no. 287, pp. 1109–1132, 2009.
- [22] K. Jacobs, X. Wang, and H. M. Wiseman, “Coherent feedback that beats all measurement-based feedback protocols,” *New Journal of Physics*, vol. 16, no. 7, p. 073036, 2014.
- [23] G. Baggio, F. Ticozzi, and L. Viola, “Quantum state preparation by controlled dissipation in finite time: From classical to quantum controllers,” in *51st IEEE Conference on Decision and Control proceedings*, 2012, pp. 1072–1077.
- [24] A. R. R. Carvalho, P. Milman, R. L. de Matos Filho, and L. Davidovich, “Decoherence, pointer engineering, and quantum state protection,” *Phys. Rev. Lett.*, vol. 86, no. 22, pp. 4988–4991, 2001.
- [25] H. Krauter, C. A. Muschik, K. Jensen, W. Wasilewski, J. M. Petersen, J. I. Cirac, and E. S. Polzik, “Entanglement generated by dissipation and steady state entanglement of two macroscopic objects,” *Phys. Rev. Lett.*, vol. 107, p. 080503, 2011.
- [26] F. Ticozzi, R. Lucchese, P. Cappellaro, and L. Viola, “Hamiltonian control of quantum dynamical semigroups: Stabilization and convergence speed,” *submitted*, 2011.
- [27] F. Ticozzi and L. Viola, “Stabilizing entangled states with quasi-local quantum dynamical semigroups,” *Phil. Trans. R. Soc. A*, vol. 370, no. 1979, pp. 5259–5269, 2012.
- [28] —, “Steady-state entanglement by engineered quasi-local markovian dissipation,” *Quantum Information and Computation*, vol. 14, no. 3–4, pp. 0265–0294, 2014.
- [29] S. Shankar, M. Hatridge, Z. Leghtas, K. M. Sliwa, A. Narla, U. Vool, S. M. Girvin, L. Frunzio, M. Mirrahimi, and M. H. Devoret, “Autonomously stabilized entanglement between two superconducting quantum bits,” *Nature*, vol. 504, no. 7480, pp. 419–422, 2013.
- [30] F. Ticozzi, S. G. Schirmer, and X. Wang, “Stabilizing quantum states by constructive design of open quantum dynamics,” *IEEE Trans. Aut. Contr.*, vol. 55, no. 12, pp. 2901–2905, 2010.
- [31] B. Kraus, S. Diehl, A. Micheli, A. Kantian, H. P. Büchler, and P. Zoller, “Preparation of entangled states by dissipative quantum markov processes,” *Phys. Rev. A*, vol. 78, no. 4, p. 042307, 2008.
- [32] P. Scaramuzza and F. Ticozzi, “Switching quantum dynamics for fast stabilization,” *Physical Review A*, no. 91, p. 062314, 2015.
- [33] J. T. Barreiro, M. Müller, P. Schindler, D. Nigg, T. Monz, M. Chwalla, M. Hennrich, C. F. Roos, P. Zoller, and R. Blatt, “An open-system quantum simulator with trapped ions,” *Nature*, vol. 470, pp. 486–491, 2011.
- [34] P. Schindler, M. Müller, D. Nigg, J. T. Barreiro, E. A. Martinez, M. Hennrich, T. Monz, S. Diehl, P. Zoller, and R. Blatt, “An open-system quantum simulator with trapped ions,” *Nature Phys.*, vol. 9, pp. 361–367, 2013.
- [35] K. Kraus, *States, Effects, and Operations: Fundamental Notions of Quantum Theory*, ser. Lecture notes in Physics. Springer-Verlag, Berlin, 1983.
- [36] M. Wolf, *Quantum Channels: A Guided Tour*. available online: <http://www-m5.ma.tum.de/foswiki/pub/M5/Allgemeines/MichaelWolf/QChannelLecture.pdf>, 2012.
- [37] G. I. Cirillo and F. Ticozzi, “Decompositions of hilbert spaces, stability analysis and convergence probabilities for discrete-time quantum dynamical semigroups,” *Journal of Physics A: Mathematical and Theoretical*, vol. 48, no. 8, p. 085302, 2015.
- [38] S. Bolognani and F. Ticozzi, “Engineering stable discrete-time quantum dynamics via a canonical QR decomposition,” *IEEE Trans. Autom. Contr.*, to appear. e-print: [arXiv:0908.2078](https://arxiv.org/abs/0908.2078), 2009.
- [39] F. Albertini and F. Ticozzi, “Discrete-time controllability for feedback quantum dynamics,” *Automatica*, vol. 47, no. 11, pp. 2451–2456, 2011.
- [40] J. von Neumann, *Functional operators*. Princeton Univ. Press Princeton, N. J., 1950, vol. 2.
- [41] I. Halperin, “The product of projection operators,” *Acta Sci. Math. (Szeged)*, vol. 23, pp. 96–99, 1962.
- [42] H. H. Bauschke and J. M. Borwein, “On projection algorithms for solving convex feasibility problems,” *SIAM Review*, vol. 38, no. 3, pp. 367–426, 1996.
- [43] R. Escalante and M. Raydan, *Alternating projection methods*. SIAM, 2011, vol. 8.
- [44] R. L. Dykstra, “An algorithm for restricted least squares regression,” *Journal of the American Statistical Association*, vol. 78, no. 384, pp. 837–842, 1983.



- [45] K. M. Grigoriadis and R. E. Skelton, "Fixed-order control design for LMI control problems using alternating projection methods," in *Decision and Control, 1994., Proceedings of the 33rd IEEE Conference on*, vol. 3. IEEE, 1994, pp. 2003–2008.
- [46] D. Drusvyatskiy, C.-K. Li, D. C. Pelejo, Y.-L. Voronin, and H. Wolkowicz, "Projection methods for quantum channel construction," *Quantum Information Processing*, vol. 14, no. 8, pp. 3075–3096, 2015.
- [47] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Prentice Hall, USA, 2002.
- [48] P. D. Johnson, F. Ticozzi, and L. Viola, "General fixed points of quasi-local frustration-free quantum semigroups: from invariance to stabilization," *Quantum Information and Computation*, vol. 16, no. 7&8, pp. 0657–0699, 2016.
- [49] D. Petz, *Quantum Information Theory and Quantum Statistics*. Springer Verlag, Berlin, 2008.
- [50] R. Blume-Kohout, H. K. Ng, D. Poulin, and L. Viola, "Information preserving structures: A general framework for quantum zero-error information," *Phys. Rev. A*, vol. 82, p. 062306, 2010.
- [51] K. R. Davidson, *C\*-Algebras by Example*. American Mathematical Soc., 1996, vol. 6.
- [52] I. Csiszár, "I-divergence geometry of probability distributions and minimization problems," *The Annals of Probability*, pp. 146–158, 1975.
- [53] L. M. Bregman, "The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming," *USSR computational mathematics and mathematical physics*, vol. 7, no. 3, pp. 200–217, 1967.
- [54] J. P. LaSalle, *The Stability and Control of Discrete Processes*, ser. Applied mathematical sciences. Springer Verlag New York, 1980, vol. 62.
- [55] L. Mazzearella, A. Sarlette, and F. Ticozzi, "Consensus for quantum networks: from symmetry to gossip iterations," *IEEE Trans. Automatic Control*, vol. 60, no. 1, pp. 158–172, 2015.
- [56] F. Ticozzi, "Symmetrizing quantum dynamics beyond gossip-type algorithms," *Automatica*, vol. 74, pp. 38–46, 2016.
- [57] D. Perez-Garcia, F. Verstraete, M. M. Wolf, and J. I. Cirac, "Matrix product state representations," *Quantum Inf. Comput.*, vol. 7, pp. 401–430, 2007.
- [58] M. J. Kastoryano and F. G. S. L. Brandao, "Quantum Gibbs samplers: the commuting case," *arXiv:1409.3435*, 2014.
- [59] M. Hein, W. Dür, J. Eisert, R. Raussendorf, M. V. den Nest, and H.-J. Briegel, "Entanglement in graph states and its applications," in *Proceedings of the International School of Physics "Enrico Fermi" on "Quantum Computers, Algorithms and Chaos"*, 2005.
- [60] B. M. Garraway, "The Dicke model in quantum optics: Dicke model revisited," *Phil. Trans. R. Soc. A*, vol. 369, p. 1137, 2011.
- [61] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, vol. I & II. Springer-Verlag, Berlin, 1979.
- [62] E. Knill, R. Laflamme, and L. Viola, "Theory of quantum error correction for general noise," *Phys. Rev. Lett.*, vol. 84, no. 11, pp. 2525–2528, 2000.
- [63] R. Blume-Kohout, H. K. Ng, D. Poulin, and L. Viola, "The structure of preserved information in quantum processes," *Phys. Rev. Lett.*, vol. 100, pp. 030 501:1–4, 2008.
- [64] F. Ticozzi and L. Viola, "Quantum information encoding, protection and correction via trace-norm isometries," *Phys. Rev. A*, vol. 81, no. 3, p. 032313, 2010.
- [65] T. Benoist, C. Pellegrini, and F. Ticozzi, "Exponential stability of subspaces for quantum stochastic master equations," *arXiv:1512.00732*, 2015.
- [66] P. D. Johnson, F. Ticozzi, and L. Viola, "Exact stabilization of entangled states in finite time by quasi-local dynamical maps," forthcoming (2017).